

# The continuity of two functions associated with a maximin problem with connected variables

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## Abstract

A function being the sum of two bilinear functions with one and the same first vector argument belonging to a polyhedron and the other two vector arguments belonging to another polyhedron is considered. It is shown that a certain minimum function of this sum and the maximin function of the sum (on the second polyhedron of connected variables) are continuous on corresponding polyhedra, which can be used in solving a maximin problem that is considered in the article.

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Let the polyhedra  $M$ ,  $H$ ,  $T$ , and  $\Omega$  be described by the following systems of linear inequalities:

$$\begin{aligned} M &= \{x \in R_+^n : Kx \geq a\}, & H &= \{y \in R_+^n : By \geq b\}, \\ T &= \{z \in R_+^n : Cz \geq c\}, & \Omega &= \{(y, z) \in H \times T : (D|G)(y, z) \geq d\}, \end{aligned}$$

where  $K$ ,  $B$ ,  $C$ ,  $D$ ,  $G$  and  $a$ ,  $b$ ,  $c$ ,  $d$  are matrices and vectors of corresponding dimensions. Further, let us consider the problem

$$(y, z, x) \in \text{Arg sup}_{y \in H} \inf_{z \in T : (y, z) \in \Omega} \inf_{x \in M} \langle x, A(y, z) \rangle, \quad (1)$$

where  $A = (A_1|A_2)$  is an  $(n \times 2n)$  matrix with real elements, and let the inequalities

$$T(y^0) = \{z \in T : (y^0, z) \in \Omega\} \neq \emptyset, \quad \forall y^0 \in H \quad (2)$$

and

$$H(z^0) = \{y \in H : (y, z^0) \in \Omega\} \neq \emptyset, \quad \forall z^0 \in T \quad (3)$$

hold.

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Problem (1)–(3) turns out to be a mathematical model for problems relating to estimating the expectation of the margin of certain parameters of various systems. In particular, as shown in [1], problem (1) is a generalization of a problem of estimating the potential margin of voters that a candidate from a major political party may receive from undecided voters in a US Federal election in a state as a result of the election campaigns conducted by all the candidates competing in the state.

**Proposition.** *The equalities*

$$\begin{aligned} \sup_{y \in H} \inf_{z \in T: (y, z) \in \Omega} \inf_{x \in M} \langle x, A(y, z) \rangle &= \max_{y \in H} \min_{z \in T: (y, z) \in \Omega} \min_{x \in M} \langle x, A(y, z) \rangle \\ &= \min_{x \in M} \max_{y \in H} \min_{z \in T: (y, z) \in \Omega} \langle x, A(y, z) \rangle \end{aligned}$$

hold.

**Proof.** 1. Let us show first that the equality

$$\sup_{y \in H} \inf_{z \in T: (y, z) \in \Omega} \inf_{x \in M} \langle x, A(y, z) \rangle = \max_{y \in H} \min_{z \in T: (y, z) \in \Omega} \min_{x \in M} \langle x, A(y, z) \rangle$$

holds. Indeed, the equality

$$\inf_{x \in M} \langle x, A(y, z) \rangle = \min_{x \in M} \langle x, A(y, z) \rangle, \quad \forall (y^0, z^0) \in \Omega$$

holds as the function  $\langle x, A(y^0, z^0) \rangle$  is a linear function on the polyhedron  $M$ . Further, the equality

$$\inf_{z \in T: (y^0, z) \in \Omega} \min_{x \in M} \langle x, A(y^0, z) \rangle = \min_{z \in T: (y^0, z) \in \Omega} \min_{x \in M} \langle x, A(y^0, z) \rangle$$

holds for any  $y^0 \in H$  as  $\langle x, A(y^0, z) \rangle = \langle x, A_1 y^0 \rangle + \langle x, A_2 z \rangle$ , the set  $T(y^0) = \{z \in T : (y^0, z) \in \Omega\} \subset T$  is a polyhedron, and by virtue of the equality

$$\min_{x \in M} \{\langle x, A_1 y^0 \rangle + \langle x, A_2 z \rangle\} = \min_{i \in \overline{1, \alpha}} \{\langle x^i, A_1 y^0 \rangle + \langle x^i, A_2 z \rangle\} \quad \forall z \in T(y^0),$$

where  $x^i \in M$ ,  $i \in \overline{1, \alpha}$  are vertices of the polyhedron  $M$ , the function

$$\min_{x \in M} \{\langle x, A_1 y^0 \rangle + \langle x, A_2 z \rangle\}$$

is continuous on  $T(y^0)$  as the minimum function of a finite number of linear functions on  $T(y^0)$  [2].

Finally, let us show that the function

$$\psi(y) = \min_{z \in T(y)} \min_{x \in M} \langle x, A(y, z) \rangle, \quad T(y) = \{z \in T : Cz \geq c, \quad Gz \geq d - Dy\}$$

is continuous on the polyhedron  $H$ .

As the equality

$$\min_{z \in T(y)} \min_{x \in M} \langle x, A(y, z) \rangle = \min_{x \in M} \min_{z \in T(y)} \langle x, A(y, z) \rangle$$

holds for any  $y \in H$ , the equality

$$\min_{x \in M} \min_{z \in T(y)} \{\langle x, A_1 y \rangle + \langle x, A_2 z \rangle\} = \min_{x \in M} \{\langle x, A_1 y \rangle + \min_{z \in T(y)} \langle x, A_2 z \rangle\}, \quad \forall y \in H$$

holds. Further, let us show that the function  $\langle x, A_1 y \rangle + \min_{z \in T(y)} \langle x, A_2 z \rangle$  is continuous on  $H$  for any  $x^* \in M$ . Indeed, for any  $x^* \in M$ , the equality

$$\begin{aligned} &|\langle x^*, A_1(y^0 + \Delta y) \rangle + \min_{z \in T(y^0 + \Delta y)} \langle x^*, A_2 z \rangle - \langle x^*, A_1 y^0 \rangle - \min_{z \in T(y^0)} \langle x^*, A_2 z \rangle| \\ &= |\langle x^*, A_1 \Delta y \rangle + \min_{z \in T(y^0 + \Delta y)} \langle x^*, A_2 z \rangle - \min_{z \in T(y^0)} \langle x^*, A_2 z \rangle| \end{aligned}$$

holds for any  $y^0 \in H$  and any  $\Delta y \in R^n : y^0 + \Delta y \in H$ . From the duality theorem of linear programming [3], it stems that the equalities

$$\min_{z \in T(y^0 + \Delta y)} \langle x^*, A_2 z \rangle = \max_{(w_1, w_2) \in \Lambda \leq x^* A_2} \langle (w_1, w_2), (c, d - D(y^0 + \Delta y)) \rangle$$

and

$$\min_{z \in T(y^0)} \langle x^*, A_2 z \rangle = \max_{(w_1, w_2) \in \Lambda \leq x^* A_2} \langle (w_1, w_2), (c, d - D y^0) \rangle,$$

where  $\Lambda = \begin{pmatrix} C \\ G \end{pmatrix}$  is a  $2n \times n$  matrix, hold so that the inequality

$$\begin{aligned} & |\langle x^*, A_1 \Delta y \rangle + \min_{z \in T(y^0 + \Delta y)} \langle x^*, A_2 z \rangle - \min_{z \in T(y^0)} \langle x^*, A_2 z \rangle| \\ & \leq |\langle x^*, A_1 \Delta y \rangle| + \left| \max_{(w_1, w_2) \in \Lambda \leq x^* A_2} \langle (w_1, w_2), (c, d - D(y^0 + \Delta y)) \rangle \right. \\ & \quad \left. - \max_{(w_1, w_2) \in \Lambda \leq x^* A_2} \langle (w_1, w_2), (c, d - D y^0) \rangle \right| \end{aligned}$$

holds. Let  $\Psi \subseteq \{(w_1, w_2) \in \Lambda \leq x^* A_2\}$  be the convex hull of all the set of the vertices of the polyhedral set  $\{(w_1, w_2) \in \Lambda \leq x^* A_2\}$ . Then the equalities

$$\max_{(w_1, w_2) \in \Lambda \leq x^* A_2} \langle (w_1, w_2), (c, d - D(y^0 + \Delta y)) \rangle = \max_{(w_1, w_2) \in \Psi} \langle (w_1, w_2), (c, d - D(y^0 + \Delta y)) \rangle$$

and

$$\max_{(w_1, w_2) \in \Lambda \leq x^* A_2} \langle (w_1, w_2), (c, d - D y^0) \rangle = \max_{(w_1, w_2) \in \Psi} \langle (w_1, w_2), (c, d - D y^0) \rangle$$

hold. As  $\Psi$  is a polyhedron, from the well-known inequality

$$|\max_{X \in Q} F_1(X) - \max_{X \in Q} F_2(X)| \leq \max_{X \in Q} |F_1(X) - F_2(X)|$$

for any functions  $F_1(X), F_2(X)$  continuous on a closed, bounded set  $Q$  [2], it stems that for any  $x^* \in M$ , the inequalities

$$\begin{aligned} & \left| \min_{z \in T(y^0 + \Delta y)} [\langle x^*, A_1(y^0 + \Delta y) \rangle + \langle x^*, A_2 z \rangle] - \min_{z \in T(y^0)} [\langle x^*, A_1 y^0 \rangle + \langle x^*, A_2 z \rangle] \right| \\ & \leq |\langle x^*, A_1 \Delta y \rangle| + \max_{(w_1, w_2) \in \Psi} |\langle (w_1, w_2), (0, -D \Delta y) \rangle| \\ & \leq \|x^* A_1\| \times \|\Delta y\| + \max_{(w_1, w_2) \in \Psi} \|w_2 D\| \times \|\Delta y\| \leq \omega \times \|\Delta y\|, \end{aligned}$$

where  $\omega \in R_+^1$ , hold. This means that the function  $\langle x^*, A_1 y \rangle + \min_{z \in T(y)} \langle x^*, A_2 z \rangle$  is continuous on  $H$  for any  $x^* \in M$ .

As for any  $y^0 \in H$ , the function  $\langle x, A_1 y^0 \rangle + \min_{z \in T(y^0)} \langle x, A_2 z \rangle$  is a bilinear function on  $M \times T(y^0)$ , this function attains its minimum on  $M \times T(y^0)$  at  $(x^{i^0}, t^{j^0}(y^0))$ , where  $x^{i^0}, i^0 \in \overline{1, \alpha}$  and  $t^{j^0}(y^0), j^0 \in \overline{1, \beta}$  are vertices of the polyhedra  $M$  and  $T(y^0)$ , respectively [4]. In particular, this means that the equality

$$\min_{x \in M} \{ \langle x, A_1 y \rangle + \min_{z \in T(y)} \langle x, A_2 z \rangle \} = \min_{i \in \overline{1, \alpha}} \{ \langle x^i, A_1 y \rangle + \min_{z \in T(y)} \langle x^i, A_2 z \rangle \}, \quad \forall y \in H,$$

where  $x^i$  are vertices of the polyhedron  $M$ , holds. As the functions

$$\langle x^i, A_1 y \rangle + \min_{z \in T(y)} \langle x^i, A_2 z \rangle, \quad i \in \overline{1, \alpha}$$

are continuous on the polyhedron  $H$ , it follows that the function

$$\min_{x \in M} \{ \langle x, A_1 y \rangle + \min_{z \in T(y)} \langle x, A_2 z \rangle \} = \min_{i \in \overline{1, \alpha}} \{ \langle x^i, A_1 y \rangle + \min_{z \in T(y)} \langle x^i, A_2 z \rangle \}$$

is also continuous on  $H$  as the minimum function of a finite number of functions continuous on  $H$  [2]. Therefore, the equality

$$\sup_{y \in H} \inf_{z \in T:(y,z) \in \Omega} \inf_{x \in M} \langle x, A(y, z) \rangle = \max_{y \in H} \min_{z \in T:(y,z) \in \Omega} \min_{x \in M} \langle x, A(y, z) \rangle$$

holds, and the sets

$$\text{Arg} \sup_{y \in H} \inf_{z \in T(y)} \inf_{x \in M} \langle x, A(y, z) \rangle$$

and

$$\text{Arg} \max_{y \in H} \min_{z \in T(y)} \min_{x \in M} \langle x, A(y, z) \rangle$$

coincide.

2. As shown earlier, the function

$$\min_{z \in T:(y,z) \in \Omega} \langle x, A(y, z) \rangle = \langle x, A_1 y \rangle + \min_{z \in T(y)} \langle x, A_2 z \rangle$$

is continuous on  $H$  for any  $x \in M$  so that the function

$$\max_{y \in H} \min_{z \in T:(y,z) \in \Omega} \langle x, A(y, z) \rangle = \langle x, A(y(x), z(x)) \rangle$$

is defined on the polyhedron  $M$ . Let us show that this function is continuous on  $M$ .

Let  $x^0 \in M$  and  $(x^0 + \Delta x) \in M$ , and let the inclusion

$$(y^0, z^0) \in \text{Arg} \max_{y \in H} \min_{z \in T(y)} \langle (x^0 + \Delta x), A(y, z) \rangle$$

hold. Then the inequality

$$\begin{aligned} & \max_{y \in H} [\langle (x^0 + \Delta x), A_1 y \rangle + \min_{z \in T(y)} \langle (x^0 + \Delta x), A_2 z \rangle] - \max_{y \in H} [\langle x^0, A_1 y \rangle + \min_{z \in T(y)} \langle x^0, A_2 z \rangle] \\ & \leq \langle (x^0 + \Delta x), A_1 y^0 \rangle + \min_{z \in T(y^0)} \langle (x^0 + \Delta x), A_2 z \rangle - \langle x^0, A_1 y^0 \rangle - \min_{z \in T(y^0)} \langle x^0, A_2 z \rangle \end{aligned}$$

holds. Therefore, by virtue of the well-known inequality

$$|\min_{z \in Q} F_1(z) - \min_{z \in Q} F_2(z)| \leq \max_{z \in Q} |F_1(z) - F_2(z)|,$$

which holds for any functions  $F_1(z)$ ,  $F_2(z)$  continuous on a closed, bounded set  $Q$  [2], the inequalities

$$\begin{aligned} & |\max_{y \in H} [\langle (x^0 + \Delta x), A_1 y \rangle + \min_{z \in T(y)} \langle (x^0 + \Delta x), A_2 z \rangle] - \max_{y \in H} [\langle x^0, A_1 y \rangle + \min_{z \in T(y)} \langle x^0, A_2 z \rangle]| \\ & \leq |\langle (x^0 + \Delta x), A_1 y^0 \rangle + \min_{z \in T(y^0)} \langle (x^0 + \Delta x), A_2 z \rangle - \langle x^0, A_1 y^0 \rangle - \min_{z \in T(y^0)} \langle x^0, A_2 z \rangle| \\ & \leq |\langle \Delta x, A_1 y^0 \rangle| + \max_{z \in T(y^0)} |\langle \Delta x, A_2 z \rangle| \leq \|\Delta x\| (\|A_1 y^0\| + \max_{z \in T(y^0)} \|A_2 z\|) \leq \|\Delta x\| \times \epsilon, \end{aligned}$$

where  $\epsilon \in R_+^1$ , also hold. This means that the function

$$\max_{y \in H} \min_{z \in T:(y,z) \in \Omega} \langle x, A(y, z) \rangle = \langle x, A(y(x), z(x)) \rangle$$

is continuous on the polyhedron  $M$  and, consequently, attains its minimum on  $M$  so that the number

$$\min_{x \in M} \max_{y \in H} \min_{z \in T:(y,z) \in \Omega} \langle x, A(y, z) \rangle$$

exists along with the number

$$\max_{y \in H} \min_{z \in T:(y,z) \in \Omega} \min_{x \in M} \langle x, A(y, z) \rangle.$$

3. Let us finally show that the equality

$$\min_{x \in M} \max_{y \in H} \min_{z \in T: (y,z) \in \Omega} \langle x, A(y, z) \rangle = \max_{y \in H} \min_{z \in T: (y,z) \in \Omega} \min_{x \in M} \langle x, A(y, z) \rangle$$

holds.

As the inequality

$$\max_{y \in H} \min_{z \in T: (y,z) \in \Omega} \langle x^0, A(y, z) \rangle \geq \min_{x \in M} \max_{y \in H} \min_{z \in T: (y,z) \in \Omega} \langle x, A(y, z) \rangle$$

holds for any  $x^0 \in M$ , and the equality

$$\max_{y \in H} \min_{z \in T: (y,z) \in \Omega} \min_{x \in M} \langle x, A(y, z) \rangle = \langle x^*, A(y^*, z^*) \rangle$$

holds for some  $x^* \in M$ ,  $(y^*, z^*) \in \Omega$ , the inequality

$$\begin{aligned} \max_{y \in H} \min_{z \in T: (y,z) \in \Omega} \min_{x \in M} \langle x, A(y, z) \rangle &= \langle x^*, A(y^*, z^*) \rangle = \max_{y \in H} \min_{z \in T: (y,z) \in \Omega} \langle x^*, A(y, z) \rangle \\ &\geq \min_{x \in M} \max_{y \in H} \min_{z \in T: (y,z) \in \Omega} \langle x, A(y, z) \rangle \end{aligned}$$

also holds. At the same time, from the equality

$$\langle x^0, A(y, z) \rangle \geq \min_{x \in M} \langle x, A(y, z) \rangle, \quad \forall (y, z) \in \Omega, \quad \forall x^0 \in M$$

and

$$\min_{z \in T: (y,z) \in \Omega} \langle x^0, A(y, z) \rangle \geq \min_{z \in T: (y,z) \in \Omega} \min_{x \in M} \langle x, A(y, z) \rangle, \quad \forall y \in H, \quad \forall x^0 \in M,$$

it follows that

$$\max_{y \in H} \min_{z \in T: (y,z) \in \Omega} \langle x^0, A(y, z) \rangle \geq \max_{y \in H} \min_{z \in T: (y,z) \in \Omega} \min_{x \in M} \langle x, A(y, z) \rangle, \quad \forall x^0 \in M$$

and, consequently, the inequality

$$\min_{x \in M} \max_{y \in H} \min_{z \in T: (y,z) \in \Omega} \langle x, A(y, z) \rangle \geq \max_{y \in H} \min_{z \in T: (y,z) \in \Omega} \min_{x \in M} \langle x, A(y, z) \rangle$$

holds (as the function

$$\max_{y \in H} \min_{z \in T: (y,z) \in \Omega} \langle x, A(y, z) \rangle$$

is continuous on  $M$ ). The proposition is proved.  $\square$

Thus, the proposition allows one to substitute the problem of finding the minimum of the maximin function on a polyhedron  $M$  for the initial problem of finding the maximin of the function  $\min_{x \in M} \langle x, A(y, z) \rangle$  on the polyhedron  $\Omega$ . Though the maximin in the latter problem is sought on a polyhedron of connected variables, the substitution may allow one to use certain optimization techniques [2,5] for solving problem (1).

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